HOMOGENEOUS DISTRIBUTIONS ON THE GRASSMANN ALGEBRA [1]

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In this paper we study homogeneous distributions on the Grassmann algebra Λ_n with n generators $e_1, \ldots, e_n$. Any element $x \in \Lambda_n$ can written as

$$x = x_0 + \sum_k x_k e_k + \sum_{l<k} x_{lk} e_l e_k + \ldots$$

with coefficients $x_0, x_k, x_{lk}, \ldots$ in $\mathbb{R}$. We determine homogeneous distributions whose supports are the whole Λ_n (up to homogeneous distributions concentrated at the hyperplane $x_0 = 0$). A description of homogeneous distributions concentrated at $x_0 = 0$ is given – for simplicity – for the case $n = 2$.

In general, the study of homogeneous distributions on algebras over $\mathbb{R}$ is an interesting and useful task. It appears, for example, when one investigates representations of matrix groups over these algebras. It has also a connection with the so-called pre-homogeneous spaces etc.

The description of homogeneous distributions on $\mathbb{R}$ and on $\mathbb{C}$ is given in [1] and [2] respectively. For Λ_1 (it is the algebra of the so-called *dual numbers*) the description was given in [4].

Let us introduce some notation and agreements.

By $\mathbb{N}$ we denote $\{0, 1, 2, \ldots\}$. The sign $\equiv$ denotes the congruence modulo 2.

We use distributions x_+^λ, x_-^λ, $|x|^\lambda$, $|x|^\lambda \mathrm{sgn}\, x$, x^{-m-1}, where $\lambda \in \mathbb{C}$, $m \in \mathbb{N}$, on the real line, see [1].

For a character of the group $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$ we shall use the following notation

$$t^{\lambda,\varepsilon} = |t|^\lambda \mathrm{sgn}^\varepsilon\, t,$$

where $t \in \mathbb{R}^*$, $\lambda \in \mathbb{C}$, $\varepsilon = 0,\ 1$. In particular, if $\lambda \in \mathbb{Z}$ and $\varepsilon \equiv \lambda$, then $t^{\lambda,\varepsilon} = t^\lambda$.

By the same symbol $x^{\lambda,\varepsilon}$ we denote the distribution $|x|^\lambda \mathrm{sgn}^\varepsilon\, x$.

We say that a distribution f on $\mathbb{R}$ is homogeneous of degree (λ, ε), where $\lambda \in \mathbb{C}$, $\varepsilon = 0,\ 1$, if

$$f(tx) = t^{\lambda,\varepsilon} f(x).$$

For given λ, ε, the space of these distributions is one-dimensional. A basis is $x^{\lambda,\varepsilon}$ excepting $\lambda = -m-1$, $\varepsilon \equiv m$, $m \in \mathbb{N}$, when a basis is $\delta^{(m)}(x)$, the m-th derivative ov the Dirac delta function $\delta(x)$ on the real line. Thus, the support of a homogeneous distribution of degree (λ, ε) is $\mathbb{R}$ excepting $\lambda = -m-1$, $\varepsilon \equiv m$, $m \in \mathbb{N}$, when it is the point 0.

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For a manifold M, let $\mathcal{D}(M)$ denote the Schwartz space of compactly supported infinitely differentiable $\mathbb{C}$-valued functions on M, with a usual topology, and $\mathcal{D}'(M)$ the space of distributions on M – of linear continuous functionals on $\mathcal{D}(M)$.

1 The Grassmann algebra and its multiplicative group

The Grassmann algebra Λ_n with n generators is (see, for example, [5]) an associative algebra over $\mathbb{R}$ generated by the unit 1 and n elements $e_1, \ldots, e_n$ with the following relations

$$1 \cdot 1 = 1,$$
$$1 \cdot e_k = e_k \cdot 1 = e_k,$$
$$e_k \cdot e_l = -e_l \cdot e_k.$$

The latter relation gives

$$e_k^2 = 0.$$

The algebra Λ_n has dimension 2^n. A basis is formed by the following elements: the unit 1 and

$$e_H = e_{i_1} e_{i_2} \ldots e_{i_p},$$

where H is the ordered set of indices: $i_1 < i_2 < \ldots < i_p$, $p = 1, \ldots, n$. The number $|H| = p$ is called the *degree* of the element e_H.

Thus, an element $x \in \Lambda_n$ can be written as

$$x = x_0 + \sum x_H e_H, \quad x_0, \, x_H \in \mathbb{R}.$$

Let $H = \{i_1 < i_2 < \ldots < i_p\}$ and $G = \{j_1 < j_2 < \ldots < j_q\}$ be two ordered sets of indices. If they do not intersect, then we denote by $\sigma(H, G)$ the number of inversions in the sequence $\{i_1, i_2, \ldots, i_p, j_1, j_2, \ldots, j_q\}$ and by $H \cdot G$ the ordered set obtained by ordering of the set $H \cup G$. In particular, if H consists of one index α, i.e. $H = \{\alpha\}$, then for brevity we write $\sigma(\alpha, G)$ and $\alpha \cdot G$ instead of $\sigma(\{\alpha\}, G)$ and $\{\alpha\} \cdot G$ respectively. Elements e_H and e_G are multiplied as follows:

$$\begin{aligned} e_H e_G &= (-1)^{\sigma(H,G)} e_{H \cdot G}, \quad H \cap G = \emptyset, \\ e_H e_G &= 0, \quad H \cap G \neq \emptyset. \end{aligned}$$

An element $x \in \Lambda_n$ is called *even* if it is a linear combination of e_H with even degrees $|H|$. The set ${}^0\Lambda_n$ of all even elements is a commutative subalgebra of Λ_n.

Let $\Lambda_n^\star$ denote the set of elements $x \in \Lambda_n$ with $x_0 \neq 0$. It is a group – the *multiplicative group* of Λ_n. Let R be the set of elements $x = x_0 \neq 0$ of Λ_n. It is a subgroup of $\Lambda_n^\star$ isomorphic to the multiplicative group $\mathbb{R}^*$. Let S be the subgroup of $\Lambda_n^\star$ consisting of elements with $x_0 = 1$. Any $x \in \Lambda_n^\star$ can be written as

$$x = x_0 \cdot \frac{x}{x_0},$$

so that $\Lambda_n^\star$ is isomorphic to $R \times S$.

The subgroup ${}^0S = S \cap {}^0\Lambda_n$ is the center and the commutant (the derived subgroup) of S. The factor group $S/{}^0S$ is isomorphic to the additive group $\mathbb{R}^n$.

2 Characters of the multiplicative group

A *character* of the group $\Lambda_n^\star$ is a continuous homomorphism of $\Lambda_n^\star$ into the multiplicative group $\mathbb{C}^*$ of complex numbers without zero.

Theorem 2.1 *Any character χ of the group $\Lambda_n^\star$ is*

$$\chi(x) = x_0^{\lambda,\varepsilon} \exp\Big\{\Big(\sum_{k=1}^{n} \mu_k x_k\Big)/x_0\Big\}, \tag{2.1}$$

where $\lambda \in \mathbb{C}$, $\varepsilon = 0, 1$, $\mu_1, \dots, \mu_n \in \mathbb{C}$.

Proof. Let $x \in \Lambda_n^\star$. We have

$$\chi(x) = \chi(x_0)\chi\Big(\frac{x}{x_0}\Big). \tag{2.2}$$

Characters of $\mathbb{R}^*$ are known, see Introduction, therefore, $\chi(x_0) = x_0^{\lambda,\varepsilon}$. The element x/x_0 belongs to S, the character χ has to be equal to 1 on the commutant ${}^0S$, therefore, it is a character of the group $\mathbb{R}^n$, so that

$$\chi\Big(\frac{x}{x_0}\Big) = \exp \sum_{k=1}^{n} \mu_k \frac{x_k}{x_0}.$$

Substituting it all into (2.2), we obtain (2.1). □

The character χ given by (2.1) will be denoted by $\chi_{\lambda,\varepsilon,\mu}$, where $\mu = (\mu_1, \dots, \mu_n) \in \mathbb{C}^n$.

3 Homogeneous distributions on the Grassmann algebra

Let us call a distribution F on Λ_n (i.e. $F \in \mathcal{D}'(\Lambda_n)$) *homogeneous of degree* $(\lambda, \varepsilon, \mu)$, where $\lambda \in \mathbb{C}$, $\varepsilon = 0, 1$, $\mu \in \mathbb{C}^n$, if

$$F(ax) = \chi_{\lambda,\varepsilon,\mu}(a)F(x), \ a \in \Lambda_n^\star. \tag{3.1}$$

The character $\chi_{\lambda,\varepsilon,\mu}$ can be considered as a distribution on Λ_n without the hyperplane $x_0 = 0$, i.e. on the set $\{x_0 \neq 0\}$. Indeed, it is a continuous function on this set. But on the whole Λ_n it can be extended only when all $\mu_1, \dots, \mu_n$ are purely imaginare, i.e. $\mu \in i\mathbb{R}^n$, see [1]. In this case it is a locally integrable function for $\operatorname{Re}\lambda > -1$ and can be continued meromorphically in λ into the whole λ-plane.

It turns out that it is a general form of homogeneous distributions whose supports are the whole Λ_n. Namely, we have the following theorem.

Theorem 3.1 *A homogeneous distribution F on Λ_n of degree $(\lambda, \varepsilon, \mu)$ with support Λ_n exists only for $\mu \in i\mathbb{R}^n$. Up to homogeneous distributions concentrated on the hyperplane $x_0 = 0$, it is $C \cdot \chi_{\lambda,\varepsilon,\mu}$ (therefore, if $\lambda = -m-1$, $m \in \mathbb{N}$, then it has to be $\varepsilon \equiv m$).*

Proof. Let a in (3.1) be such that $a_0 > 0$. Differentiating (3.1) with respect to a_0, a_H and set $a_0 = 1$, $a_H = 0$, we obtain the following system:

$$x_0 \frac{\partial F}{\partial x_0} + \sum_G x_G \frac{\partial F}{\partial x_G} = \lambda F, \tag{3.2}$$

$$x_0 \frac{\partial F}{\partial x_\alpha} + \sum_G (-1)^{\sigma(\alpha,G)} x_G \frac{\partial F}{\partial x_{\alpha \cdot G}} = \mu_\alpha F, \ \alpha = 1, \ldots, n, \tag{3.3}$$

$$x_0 \frac{\partial F}{\partial x_H} + \sum_G (-1)^{\sigma(H,G)} x_G \frac{\partial F}{\partial x_{H \cdot G}} = 0, \ |H| > 1. \tag{3.4}$$

First consider F on the set $\{x_0 \neq 0\}$. Then from equations (3.4) we obtain that F does not depend on x_H, $|H| > 1$, i.e. $F = F(x_0, x_1, \ldots, x_n)$. Then equations (3.3) become

$$x_0 \frac{\partial F}{\partial x_\alpha} = \mu_\alpha F, \ \alpha = 1, \ldots, n.$$

It gives

$$F = A(x_0) \exp\Big\{\Big(\sum_{\alpha=1}^n \mu_\alpha x_\alpha\Big)/x_0\Big\}.$$

Then equation (3.2) gives the following equation for A:

$$x_0 \frac{dA}{dx_\alpha} = \lambda A,$$

whence

$$A = C_1 (x_0)_+^\lambda + C_2 (x_0)_-^\lambda.$$

Equation (3.1) with $a = -1$ gives that $A(x_0)$ has parity ε, so that $A = C \cdot x_0^{\lambda,\varepsilon}$.

Thus, on the set $\{x_0 \neq 0\}$ we have

$$\begin{aligned} F(x) &= C \cdot x_0^{\lambda,\varepsilon} \exp\Big\{\Big(\sum_{k=1}^n \mu_k x_k\Big)/x_0\Big\} \\ &= C \cdot \chi_{\lambda,\varepsilon,\mu}(x). \end{aligned}$$

with arbitrary $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{C}^n$.

To the whole Λ_n this distribution can be extended only when $\mu \in i\mathbb{R}^n$. □

Now let us find homogeneous distributions concentrated at the hyperplane $x_0 = 0$. For simplicity we restrict ourselves to the case $n = 2$.

For brevity we denote $e_1 e_2 = e_3$ and $x_{12} = x_3$, so that any $x \in \Lambda_2$ is

$$x = x_0 + x_1 e_1 + x_2 e_2 + x_3 e_3.$$

Theorem 3.2 *A homogeneous distribution on Λ_2 of degree $(\lambda, \varepsilon, \mu)$, where $\mu = (\mu_1, \mu_2) \in \mathbb{C}^2$, concentrated at the hyperplane $x_0 = 0$, exists only for $\mu \in i\mathbb{R}^2$, i.e. $\mu_1 = i\nu_1$, $\mu_2 = i\nu_2$, where $\nu = (\nu_1, \nu_2) \in \mathbb{R}^2$. Let us denote $s = \nu_1^2 + \nu_2^2$.*

Let $s \neq 0$. Then in variables x_0, u_1, u_2, x_3, where

$$u_1 = \nu_1 x_1 + \nu_2 x_2, \tag{3.5}$$

$$u_2 = -\nu_2 x_1 + \nu_1 x_2, \tag{3.6}$$

any homogeneous distribution F on Λ_2 of degree $(\lambda, \varepsilon, i\nu)$, concentrated at the hyperplane $x_0 = 0$, is

$$F(x) = C \cdot \delta(x_0)\delta(u_1)u_2^{\lambda+2,\varepsilon} \exp(isx_3/u_2) \tag{3.7}$$

with $\varepsilon \equiv \lambda$ for $\lambda = -3, -4, \ldots$.

Let $s = 0$, i.e. $\nu_1 = \nu_2 = 0$. Then any homogeneous distribution F on Λ_2 of degree $(\lambda, \varepsilon, 0)$, concentrated at the hyperplane $x_0 = 0$, is

$$F = \delta(x_0, x_1, x_2)\Phi(x_3), \ \lambda \neq -2, -3, -4, \ldots, \tag{3.8}$$

$$F = \delta(x_0, x_1, x_2)\Phi(x_3) + C\delta^{(-\lambda-1)}(x_0), \ \lambda = -3, -4, \ldots, \tag{3.9}$$

$$\begin{aligned} F &= \delta(x_0, x_1, x_2)\Phi(x_3) \\ &+ \delta(x_0)\Big[B_0(x_1, x_2) + B_1(x_1, x_2)x_3\Big] + A_1(x_1, x_2)\delta'(x_3), \ \lambda = -2, \end{aligned} \tag{3.10}$$

where Φ is a homogeneous distribution on $\mathbb{R}$ of degree $(\lambda + 3, \varepsilon)$, the number C in (3.9) is equal to 0 for $\varepsilon \equiv \lambda$, the distributions B_0, B_1, A_1 on $\mathbb{R}^2$ are homogeneous of degrees $(-1, \varepsilon)$, $(-2, \varepsilon + 1)$, $(0, \varepsilon + 1)$, respectively, and

$$\frac{\partial A_1}{\partial x_1} - x_2 B_1 = 0, \tag{3.11}$$

$$\frac{\partial A_1}{\partial x_2} + x_1 B_1 = 0. \tag{3.12}$$

Proof. Now (for $n = 2$) the system (3.2) – (3.4) is

$$x_0\frac{\partial F}{\partial x_0} + x_1\frac{\partial F}{\partial x_1} + x_2\frac{\partial F}{\partial x_2} + x_3\frac{\partial F}{\partial x_3} = \lambda F, \tag{3.13}$$

$$x_0\frac{\partial F}{\partial x_1} + x_2\frac{\partial F}{\partial x_3} = \mu_1 F, \tag{3.14}$$

$$x_0\frac{\partial F}{\partial x_2} - x_1\frac{\partial F}{\partial x_3} = \mu_2 F, \tag{3.15}$$

$$x_0\frac{\partial F}{\partial x_3} = 0. \tag{3.16}$$

We want to find solutions of this system concentrated at the hyperplane $x_0 = 0$. First consider functions in $\mathcal{D}(\Lambda_2) = \mathcal{D}(\mathbb{R}^4)$ whose supports belong to the ball $\|x\| < M$, where

$$\|x\|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2.$$

At such functions any distribution F concentrated at $x_0 = 0$ has the form, see [3]:

$$F = \sum_{j=0}^{N} \delta^{(j)}(x_0)A_j(x_1, x_2, x_3),$$

where A_j are distributions on $\mathbb{R}^3$. Since

$$x_0\delta^{(j)}(x_0) = -j\delta^{(j-1)}(x_0),$$

equation (3.13) gives

$$\sum_{j=0}^{N} \delta^{(j)}(x_0)\Big\{ - (j+1)A_j + DA_j\Big\} = \sum_{j=0}^{N} \lambda\delta^{(j)}(x_0)A_j,$$

where

$$D = x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3}.$$

Since the delta function and its derivatives are linearly independent, it gives

$$DA_j = (\lambda + j + 1)A_j, \quad j = 0, 1, \ldots, N. \tag{3.17}$$

Equation (3.16) gives

$$\sum_{j=1}^{N} j\delta^{(j-1)}(x_0)\frac{\partial A_j}{\partial x_3} = 0,$$

whence

$$\frac{\partial A_j}{\partial x_3} = 0, \quad j = 1, \ldots, N, \tag{3.18}$$

so that A_j for $j = 1, \ldots, N$ does not depend on x_3: $A_j = A_j(x_1, x_2)$. After that equations (3.14) and (3.15) become as follows:

$$\delta(x_0)\left[\mu_1 A_0 + \frac{\partial A_1}{\partial x_1} - x_2\frac{\partial A_0}{\partial x_3}\right] + \sum_{j=1}^{N-1}\delta^{(k)}(x_0)\left[\mu_1 A_k + (k+1)\frac{\partial A_{k+1}}{\partial x_1}\right]$$
$$+\delta^{(N)}(x_0)\mu_1 A_N = 0, \tag{3.19}$$

$$\delta(x_0)\left[\mu_2 A_0 + \frac{\partial A_1}{\partial x_2} + x_1\frac{\partial A_0}{\partial x_3}\right] + \sum_{j=1}^{N-1}\delta^{(k)}(x_0)\left[\mu_2 A_k + (k+1)\frac{\partial A_{k+1}}{\partial x_2}\right]$$
$$+\delta^{(N)}(x_0)\mu_2 A_N = 0. \tag{3.20}$$

If at least one of μ_1, μ_2 is not equal to 0, then (3.19) and (3.20) give that all $A_1, \ldots, A_N$ are equal to zero:

$$A_1 = 0, \ldots, A_N = 0,$$

and A_0 satisfy the following two equations:

$$\mu_1 A_0 - x_2\frac{\partial A_0}{\partial x_3} = 0, \tag{3.21}$$

$$\mu_2 A_0 + x_1\frac{\partial A_0}{\partial x_3} = 0. \tag{3.22}$$

Besides it, for A_0 we have also equation (3.17) with $j = 0$:

$$DA_0 = (\lambda + 1)A_0.$$

Multiplying equations (3.21) and (3.22) by x_1 and x_2 respectively and summing up, we obtain

$$(\mu_1 x_1 + \mu_2 x_2)A_0 = 0. \tag{3.23}$$

Equation (3.21) with $x_2 \neq 0$ gives that $A_0 = K(x_1, x_2)\exp(\mu_1 x_3/x_2)$. To all x_2 it can be extended only when $\mu_1 = i\nu_1$, $\nu_1 \in \mathbb{R}$. Similarly, equation (3.22) gives $\mu_2 = i\nu_2$, $\nu_2 \in \mathbb{R}$.

Let us pass from variables x_1 and x_2 to variables u_1 and u_2 by formulae (3.5), (3.6). Then equations (3.21) and (3.23) become respectively

$$isA_0 - u_2\frac{\partial A_0}{\partial x_3} = 0,$$
$$u_1 A_0 = 0. \tag{3.24}$$

As to equation (3.22), it becomes $u_1 \cdot (\partial A_0/\partial x_3) = 0$, a consequence of (3.24).

Equation (3.24) gives

$$A_0 = \delta(u_1)K(u_2, x_3). \tag{3.25}$$

For this distribution K we obtain equations

$$sK = u_2 \frac{\partial K}{\partial x_3},$$
$$u_2 \frac{\partial K}{\partial u_2} + x_3 \frac{\partial K}{\partial x_3} = (\lambda + 2)K.$$

Any solution of this system is

$$K = L(u_2)\exp(isx_3/u_2), \tag{3.26}$$

where L is a distribution on $\mathbb{R}$ of homogeneity $\lambda + 2$ such that (3.26) makes sense. Taking into account parity ε of F, we obtain

$$K = C \cdot u_2^{\lambda+2,\varepsilon} \exp(isx_3/u_2).$$

Together with (3.25) it gives (3.7).

Now let $\mu_1 = \mu_2 = 0$. Then from (3.19), (3.20) we obtain

$$\frac{\partial A_2}{\partial x_1} = \ldots = \frac{\partial A_N}{\partial x_1} = 0, \tag{3.27}$$
$$\frac{\partial A_2}{\partial x_2} = \ldots = \frac{\partial A_N}{\partial x_2} = 0, \tag{3.28}$$
$$\frac{\partial A_1}{\partial x_1} - x_2 \frac{\partial A_0}{\partial x_3} = 0, \tag{3.29}$$
$$\frac{\partial A_1}{\partial x_2} + x_1 \frac{\partial A_2}{\partial x_3} = 0. \tag{3.30}$$

We see from (3.18), (3.27) and (3.28) that $A_2, \ldots, A_N$ are constants: $A_k = C_k$, $k = 2, \ldots, N$. On the other hand, by (3.17) A_k is homogeneous of degree $\lambda + k + 1$. Therefore, if $\lambda \neq -k-1$, where $k = 2, 3, \ldots$, then all A_k are equal to zero. And if $\lambda = -k-1$, then all A_j are equal to zero, except $A_k = C_k$.

Now let us analyze system (3.29), (3.30) for A_0, A_1. Moreover, we have to add to it an equation from (3.18) and homogeneity conditions from (3.17):

$$\frac{\partial A_1}{\partial x_3} = 0, \quad DA_0 = (\lambda + 1)A_0, \quad DA_1 = (\lambda + 2)A_1. \tag{3.31}$$

Let us multiply (3.29) by x_1 and (3.30) by x_2 and sum up. Taking into account the first equation in (3.31), we obtain $DA_1 = 0$. Therefore, if $\lambda \neq -2$, then $A_1 = 0$ and system (3.29), (3.30) becomes

$$x_2 \frac{\partial A_0}{\partial x_3} = 0, \quad x_1 \frac{\partial A_0}{\partial x_3} = 0,$$

whence

$$A_0 = \delta(x_1)\delta(x_2)\Phi(x_3).$$

Substituting it into the second equation in (3.31), we get $d\Phi/dx_3 = (\lambda+3)\Phi$, so that Φ is a homogeneous distribution on $\mathbb{R}$ of degree $\lambda+3$. Taking into account parity ε of F, we obtain (3.8), (3.9).

It remains to consider the case $\lambda = -2$ for system (3.29), (3.30). Let us differentiate (3.29) and (3.30) with respect to x_3 and take into account the first equation in (3.31), we get

$$x_2 \frac{\partial^2 A_0}{\partial x_3^2} = 0, \quad x_1 \frac{\partial^2 A_0}{\partial x_3^2} = 0.$$

It gives

$$A_0 = B_0 + B_1 x_3 + \delta(x_1)\delta(x_2)\psi(x_3),$$

where $B_0 = B_0(x_1, x_2)$ and $B_1 = B_1(x_1, x_2)$. Substituting it into (3.29), (3.30), we obtain

$$\frac{\partial A_1}{\partial x_1} - x_2 B_1 = 0,$$
$$\frac{\partial A_1}{\partial x_2} + x_1 B_1 = 0.$$

It is just system (3.11), (3.12). Therefore, B_1 is homogeneous of degree (-2), hence B_1 and ψ are homogeneous of degrees (-1) and 1 respectively. Taking into account parity ε of F, we obtain (3.10).

Thus, we found homogeneous distributions F on test functions with supports in the ball $\|x\| < M$. We see that these distributions do not depend on M. $\square$

REFERENCES

1. I. M. Gelfand, G. E. Shilov. Generalized Functions and Operations on Them. M.: Fizmatgiz, 1958.

2. I. M. Gelfand, M. I. Graev, N. Ya. Vilenkin. Integral Geometry and Representation Theory. M.: Fizmatgiz, 1962.

3. G. E. Shilov. Mathematical Analysis (the second special course). M.: Nauka, 1965.

4. L. M. Molchanova. Homogeneous generalized functions on the plane of dual variable. Vestnik Tambov Univ., 2002, tom 7, issue 1, 54–55.

5. B. L. van der Waerden. Algebra. M.: Nauka, 1978.